

THERE ARE ONLY FINITELY MANY 3-SUPERBRIDGE KNOTS

CHOON BAE JEON AND GYO TAEK JIN

ABSTRACT. Although there are infinitely many knots with superbridge index n for every even integer $n \geq 4$, there are only finitely many knots with superbridge index 3.

1. INTRODUCTION

Throughout this article a *knot* is a piecewise smooth simple closed curve embedded in the three dimensional Euclidean space \mathbb{R}^3 . For a knot K , its equivalence class, under piecewise smooth homeomorphisms of \mathbb{R}^3 mapping one knot onto another, will be referred to as the *knot type* of K and denoted by $[K]$.

In 1954, Schubert introduced the *bridge index* of knots [16]. In a knot diagram, maximal overpasses are called *bridges*. Figure 1 shows a knot diagram with seven bridges which are drawn with thick arcs. The bridge index of a knot is defined to be the minimum number of bridges in all the possible diagrams of knots in its knot type. An equivalent definition can be given in the following way. Given a knot K and a unit vector \vec{v} in \mathbb{R}^3 , we define $b_{\vec{v}}(K)$ as the number of connected components of the preimage of the set of local maximum values of the orthogonal projection $K \rightarrow \mathbb{R}\vec{v}$. Figure 2 illustrates an example. The *bridge number* of K is defined by the formula

$$b(K) = \min_{\|\vec{v}\|=1} b_{\vec{v}}(K).$$

It is known that the bridge index can be defined by the formula

$$b[K] = \min_{K' \in [K]} b(K') = \min_{K' \in [K]} \min_{\|\vec{v}\|=1} b_{\vec{v}}(K').$$

In 1987, Kuiper modified the alternative definition of bridge index to define another knot invariant called *superbridge index* [9]. Given a knot K , the *superbridge*

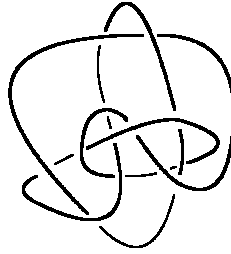
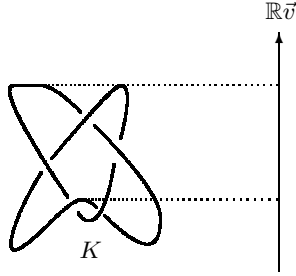


FIGURE 1. Bridges—maximal overpasses

FIGURE 2. $b_{\vec{v}}(K) = 3$

number of K is defined by

$$s(K) = \max_{\|\vec{v}\|=1} b_{\vec{v}}(K)$$

and the *superbridge index* of K by

$$s[K] = \min_{K' \in [K]} s(K') = \min_{K' \in [K]} \max_{\|\vec{v}\|=1} b_{\vec{v}}(K').$$

He used Milnor's total curvature [12] to prove that any nontrivial knot K satisfies the inequality:

$$(1) \quad b[K] < s[K]$$

He computed the superbridge index for all torus knots.

Theorem 1 (Kuiper). *For any two coprime integers p and q , satisfying $2 \leq p < q$, the superbridge index of the torus knot of type (p, q) is $\min\{2p, q\}$.*

2. ODD-SUPERBRIDGE KNOTS

As the knots having bridge index n are referred to as n -bridge knots, we will call the knots with superbridge index n as n -superbridge knots.

Because nontrivial knots have bridge index at least 2, the inequality (1) implies that nontrivial knots have superbridge index at least 3. By the same reason, 3-superbridge knots are 2-bridge knots, in particular, prime knots. According to Theorem 1, trefoil knot is the only torus knot with superbridge index 3. Figure eight knot is also a 3-superbridge knot [7, 18]. No other 3-superbridge knots are known yet. Our main theorem asserts that there are only finitely many 3-superbridge knots.

Theorem 2. *There are only finitely many 3-superbridge knots.*

By Theorem 1, we know that the torus knot of type $(n, nk + 1)$ has superbridge index $2n$, for $n \geq 2$ and $k \geq 2$. Therefore, for any even number $2n \geq 4$, there are infinitely many $2n$ -superbridge knots.

Because it is natural to expect that more knotting would increase the superbridge number, we expect that there are infinitely many n -superbridge knots for any positive integer $n \geq 4$.

Conjecture 1. *There are infinitely many $(2n - 1)$ -superbridge knots for any positive integer $n \geq 3$.*

A result in [8] implies that a connected sum of any torus knot K with a trefoil knot T satisfies the inequality $s[K\sharp T] \leq s[K] + 1$. As it is generally expected that the superbridge index of a composite knot would be bigger than that of any of the factor knots, which is true for bridge index, we make Conjecture 2 which implies Conjecture 1. This conjecture is valid if K is a trefoil knot or figure eight knot.

Conjecture 2. *Every nontrivial knot K satisfies*

$$s[K\sharp T] = s[K] + 1$$

where T is a trefoil knot.

By Theorem 1, we know that the torus knot of type $(n, 2n - 1)$ is a $(2n - 1)$ -superbridge knot. This knot is the closure of the n -braid $(\sigma_1 \sigma_2 \cdots \sigma_{n-1})^{2n-1}$. On the other hand, the torus knot of type $(2, 2k - 1)$ is the closure of the 2-braid σ_1^{2k-1} and has superbridge index 4 if $k \geq 3$. For these torus knots, inserting a full twist σ_1^2 does not increase the superbridge index. This fact encourages us to consider Conjecture 3 which also implies Conjecture 1.

Conjecture 3. *For $n \geq 3$ and $k \geq 0$, the closure of the n -braid*

$$(2) \quad \sigma_1^{2k} (\sigma_1 \sigma_2 \cdots \sigma_{n-1})^{2n-1}$$

is a $(2n - 1)$ -superbridge knot.

A theorem of Stallings [17] implies that the closure of the braid in (2) for $k \geq 0$ is a fibred knot with the fibre surface obtained by Seifert's algorithm on the closed braid diagram. This surface is the one with minimal genus, which is $(n - 1)^2 + k$. Therefore for each n , such knots are all distinct. Notice that the braid (2) is positive and the diagram of its closure is visually prime. According to Cromwell [4], they are all prime knots. The primeness of these knots makes Conjecture 3 more interesting than Conjecture 2.

The second author would like to thank Paul Melvin for a discussion which inspired Conjecture 2 and to thank Dale Rolfsen for bringing Stallings' theorem to his attention.

3. PROOF OF THEOREM 2

Our proof of Theorem 2 requires two main tools. The first is Lemma 3 and the second is *quadriseccant* which is a straight line intersecting a knot at four distinct points. According to [13, 14], every nontrivial knot has a quadriseccant.

Lemma 3. *Given a knot K , let K' be a knot obtained by replacing a subarc of K with a straight line segment joining the end points of the subarc. Then $s(K) \geq s(K')$.*

PROOF: Given a unit vector \vec{v} , let $g: (-1, 2) \rightarrow \mathbb{R}\vec{v}$ be a parametrization of the orthogonal projection of an open neighborhood of the subarc into $\mathbb{R}\vec{v}$, where the subarc corresponds to the closed interval $[0, 1]$. Then the projection of a neighborhood of the straight line segment in K' can be parametrized by

$$g'(t) = \begin{cases} (1 - t)g(0) + tg(1) & \text{if } t \in [0, 1] \\ g(t) & \text{if } t \in (-1, 0] \cup [1, 2). \end{cases}$$

Since g' has no more local maxima than g , we have $b_{\vec{v}}(K) \geq b_{\vec{v}}(K')$ for any \vec{v} . Therefore $s(K) \geq s(K')$. \square

Let K be a 3-superbridge knot with superbridge number 3, namely, $s[K] = s(K) = 3$, and let \mathcal{Q} be a quadrisecant of K . Then $K - \mathcal{Q}$ consists of four disjoint open arcs l_1, l_2, l_3 and l_4 . Let \bar{l}_i and \tilde{l}_i denote $\pi(l_i)$ and $\pi(\mathcal{Q} \cup l_i)$, respectively, where $\pi: \mathbb{R}^3 \rightarrow \mathcal{Q}^\perp$ is the orthogonal projection of \mathbb{R}^3 onto a plane \mathcal{Q}^\perp perpendicular to the quadrisecant. Applying Lemma 3 wherever needed, we may assume that the only singular points of $\pi(K)$ are a set of finitely many transversal double points together with a quadruple point $\pi(\mathcal{Q})$. For every open subarc l of K , write $b_{\vec{v}}(K | l)$ for the number of local maxima of $K \rightarrow \mathbb{R}\vec{v}$ on l . Since each \tilde{l}_i is a closed loop in \mathcal{Q}^\perp , we must have

$$(3) \quad b_{\vec{v}}(K | l_i) \geq 1 \quad \text{or} \quad b_{-\vec{v}}(K | l_i) \geq 1$$

for every unit vector $\vec{v} \in \mathcal{Q}^\perp$.

For a straight line ρ in \mathcal{Q}^\perp , let \vec{v}_ρ denote a unit vector in \mathcal{Q}^\perp perpendicular to ρ .

Sublemma 1. *We may assume that \bar{l}_i has no self-crossings, for each $i = 1, 2, 3, 4$.*

PROOF: Suppose \bar{l}_i has a self-crossing. Then we can choose a loop λ of \bar{l}_i which is *minimal* in the sense that no proper subarc of λ is another loop. Then λ bounds an open disk δ in \mathcal{Q}^\perp .

If $\pi(K) \cap \delta = \emptyset$, we can eliminate this loop together with its crossing by a move as described in Lemma 3 without changing the knot type.

If \bar{l}_i passes through δ , then among the half-lines starting from $\pi(\mathcal{Q})$ and passing through δ , we are able to find one, say ρ , which meets \bar{l}_i at least three times. Then we have

$$(4) \quad b_{\vec{v}_\rho}(K | l_i) \geq 2 \quad \text{and} \quad b_{-\vec{v}_\rho}(K | l_i) \geq 2.$$

This, together with the fact (3), implies

$$(5) \quad b_{\vec{v}_\rho}(K) \geq \sum_{1 \leq j \leq 4} b_{\vec{v}_\rho}(K | l_j) \geq 4 \quad \text{or} \quad b_{-\vec{v}_\rho}(K) \geq \sum_{1 \leq j \leq 4} b_{-\vec{v}_\rho}(K | l_j) \geq 4$$

which contradicts $s(K) = 3$.

If \bar{l}_j passes through δ , for some $j \neq i$, then among the half-lines starting from $\pi(\mathcal{Q})$ and passing through δ , we are able to find one, say ρ , which crosses \bar{l}_j . Then, for $\vec{w} = \vec{v}_\rho$ or $\vec{w} = -\vec{v}_\rho$, we have

$$(6) \quad b_{\vec{w}}(K | l_i) \geq 2, \quad b_{-\vec{w}}(K | l_i) \geq 1, \quad b_{\vec{w}}(K | l_j) \geq 1, \quad b_{-\vec{w}}(K | l_j) \geq 1.$$

This, together with the fact (3), implies (5) which contradicts $s(K) = 3$. \square

Sublemma 2. *We may assume that each \tilde{l}_i bounds an open disk δ_i in \mathcal{Q}^\perp which is star-shaped with respect to $\pi(\mathcal{Q})$.*

PROOF: By Sublemma 1, we know that \tilde{l}_i bounds an open disk δ_i in \mathcal{Q}^\perp . If δ_i is not star-shaped, there exists a half-line ρ in \mathcal{Q}^\perp starting at $\pi(\mathcal{Q})$ and meeting \bar{l}_i more than once. If ρ meets \bar{l}_i at three or more points, then the condition (4) holds. Therefore we reach the same contradiction as in (5). Suppose ρ meets \bar{l}_i at two points. Then there exist two open disks R and S bounded by ρ and \bar{l}_i as in Figure 3(i). If \bar{l}_j meets $R \cup S$, there is a half line ρ' starting from $\pi(\mathcal{Q})$ crossing \bar{l}_j

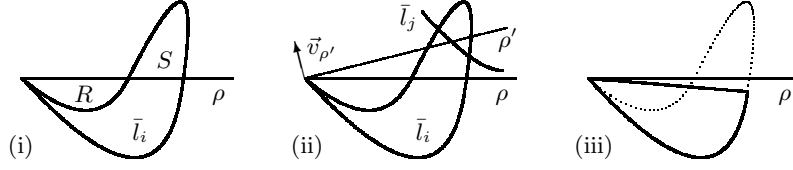


FIGURE 3.

at a point in $R \cup S$ as indicated by Figure 3(ii). Then, for $\vec{w} = \vec{v}_{\rho'}$ or $\vec{w} = -\vec{v}_{\rho'}$, the condition (6) holds. This leads to the same contradiction as in (5). If there are no arcs of $\pi(K)$ inside $R \cup S$, we can straighten a part of \bar{l}_i as in Figure 3(iii) without changing the knot type. By Lemma 3, this move doesn't increase the superbridge number. Since $s[K] = s(K) = 3$, this move cannot decrease the superbridge number either. Only finitely many of such modifications are necessary to deform δ_i into a region star-shaped with respect to $\pi(\mathcal{Q})$. \square

Sublemma 3. *None of the following conditions hold when h, i, j, k are distinct elements of $\{1, 2, 3, 4\}$.*

- (7) $\delta_i \cap \delta_j \cap \delta_k \neq \emptyset$
- (8) $\delta_i \cap \delta_j \neq \emptyset$ and $\delta_h \cap \delta_k \neq \emptyset$
- (9) $\delta_i \cap \delta_j \neq \emptyset$, $\delta_i \cap \delta_k \neq \emptyset$ and $\delta_i \cap \delta_h \neq \emptyset$

PROOF: For each of the three conditions, we will choose a line ρ in \mathcal{Q}^\perp which meets $\pi(K)$ at least eight times. Then we must have $b_{\vec{v}_\rho}(K) \geq 4$, which contradicts $s(K) = 3$.

CONDITION (7): If this condition is true, we can choose two points $P_1 \in \delta_i \cap \delta_j \cap \delta_k$ and $P_2 \in \delta_h$ so that the straight line ρ joining P_1 and P_2 does not pass through $\pi(\mathcal{Q})$. Then each \bar{l}_a crosses ρ at least twice, for $a = h, i, j, k$.

CONDITION (8): If this condition is true, we can choose two points $P_1 \in \delta_i \cap \delta_j$ and $P_2 \in \delta_h \cap \delta_k$ so that the straight line ρ joining P_1 and P_2 does not pass through $\pi(\mathcal{Q})$. Then again, each \bar{l}_a crosses ρ at least twice, for $a = h, i, j, k$.

CONDITION (9): If this condition is true, we can choose three points $P_a \in \delta_i \cap \delta_a$ for $a = j, k, h$, so that the three straight lines determined by pairs of P_a 's do not pass through $\pi(\mathcal{Q})$. Since (7) cannot occur, every edge of the triangle $\triangle P_j P_k P_h$ meets $\pi(K)$ in even number of times. There are two subcases to consider:

SUBCASE (9.1): If $\pi(\mathcal{Q})$ is contained inside $\triangle P_j P_k P_h$, the boundary of this triangle meets $\pi(K)$ at least eight times. Therefore there is an edge, say $\overline{P_j P_k}$, meeting $\pi(K)$ at least four times. Let ρ be the extension of $P_j P_k$.

SUBCASE (9.2): If $\pi(\mathcal{Q})$ is contained outside of $\triangle P_j P_k P_h$, there is one vertex of $\triangle P_j P_k P_h$, say P_h , such that the straight line segment joining P_h and $\pi(\mathcal{Q})$ crosses the edge $\overline{P_j P_k}$. Since (7) cannot occur, the edge $\overline{P_j P_k}$ meets $\pi(K)$ at least four times. Let ρ be the extension of $P_j P_k$.

For the above two subcases, ρ meets $\pi(K)$ at least twice on either side of the extension.

Consequently, ρ meets $\pi(K)$ at least eight times as required. \square

Sublemma 4. *We may assume that the only crossings in $\bar{l}_i \cup \bar{l}_j$ are a set of finitely many consecutive half twists.*

PROOF: It follows from Sublemmas 1–3 that if a half-line starting from $\pi(\mathcal{Q})$ meets \bar{l}_i and \bar{l}_j , then it meets each of them exactly once and no other \bar{l}_a 's. Notice that the region $\delta_i \cap \delta_j$ is also an open disk which is star-shaped with respect to $\pi(\mathcal{Q})$ such that the only singular points along $\partial(\delta_i \cap \delta_j)$ are the quadruple point $\pi(\mathcal{Q})$ and the half-twists. \square

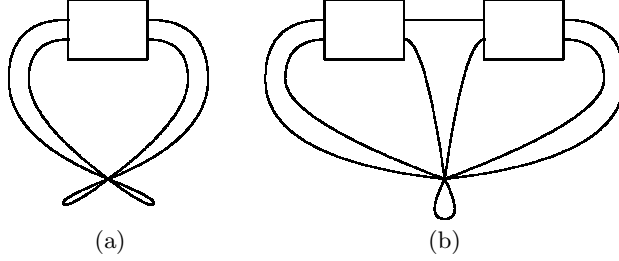


FIGURE 4.

The next sublemma easily follows from Sublemmas 2–4.

Sublemma 5. *We may assume that $\pi(K)$ is as in Figure 4 up to planar isotopies of \mathcal{Q}^\perp , where each rectangle contains a pair of parallel arcs or a pair of arcs with finitely many half-twists.*

Suppose that $\delta_i \cap \delta_j \neq \emptyset$ and that none of δ_i and δ_j contains the other completely. Consider a connected component δ of $\delta_i - (\delta_j \cup l_j)$ which does not meet $\pi(K)$. By Sublemma 4, we easily see that $\partial\delta$ has only two singular points of $\pi(K)$. Such a region will be referred to as a *crescent* and the two singular points the *ends* of the crescent. It is possible for a crescent to have the quadruple point $\pi(\mathcal{Q})$ as one of its ends. In this case, it is possible to have a *loop-crescent* which is bounded by one loop which passes through $\pi(\mathcal{Q})$ and is the projection of one subarc of K .

Sublemma 6. *We may assume that every crescent which is not a loop-crescent is alternating, in the sense that, if one of the two arcs on the boundary of the crescent passes over the other at one end then it passes under the other at the other end. We may also assume that no crescent is a loop-crescent.*

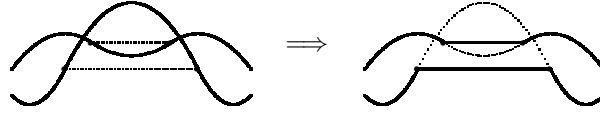


FIGURE 5. Non-alternating crescent

PROOF: If a crescent which is away from $\pi(\mathcal{Q})$ is non-alternating, we can remove the two crossings at its ends by straightening two arcs as shown in Figure 5. Since this is a second Reidemeister move, the knot type does not change. Again by Lemma 3, the superbridge number is unchanged. For a non-alternating crescent whose one end is at $\pi(\mathcal{Q})$, a similar process eliminates the crossing at the other end if it is not a loop-crescent.

If there is a loop-crescent, we may straighten a small subarc of the loop near $\pi(\mathcal{Q})$ without changing the knot type and the superbridge number. Then \mathcal{Q} becomes a *trisecant* of the new knot again denoted by K . Since K is nontrivial, the new knot must be obtained from Figure 4(b), and hence its projection must be as in Figure 6(a), after straightening out any unnecessary half-twists. On the other hand, the projection of a cylindrical neighborhood of a trisecant of an arbitrary knot has five possible patterns as shown in Figure 6(b) where all three arcs are smooth, up to small perturbations and planar isotopies. All the combinations of Figure 6(a) and one of Figure 6(b) making a nontrivial knot give rise to torus knots. Because trefoil knots are the only 3-superbridge torus knots, we can exclude the case of loop-crescents. \square

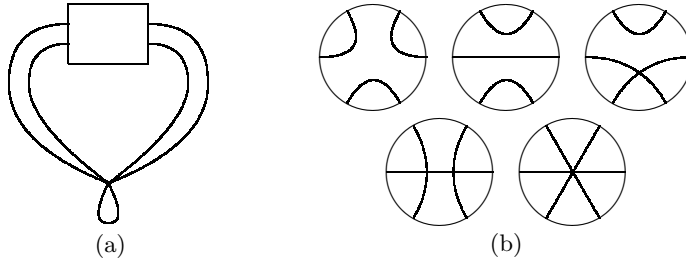


FIGURE 6.

Sublemma 7. *We may assume that no two arcs \bar{l}_i and \bar{l}_j can meet more than three times.*

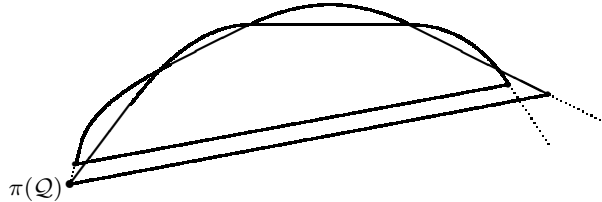
PROOF: Suppose the two arcs \bar{l}_i and \bar{l}_j meet at least four times. We now choose an orientation of the knot K . This orientation induces an orientation on $\pi(K)$ and its strings \bar{l}_i and \bar{l}_j . Now there are two cases to consider:

CASE 1. *The orientations of \bar{l}_i and \bar{l}_j are consistent around the boundary of $\delta_i \cap \delta_j$.*

— Notice that \bar{l}_i and \bar{l}_j create at least four consecutive crescents such that one end of the first is at $\pi(\mathcal{Q})$ and no ends of the last is at $\pi(\mathcal{Q})$. Straightening the two complementary subarcs of the two subarcs around the four crescents between \bar{l}_i and \bar{l}_j , we are able to obtain a five crossing knot T as depicted in Figure 7. Sublemma 6 guarantees that T is alternating and hence is a torus knot of type $(2, 5)$ which has superbridge index 4. By Lemma 3, we get a contradiction $4 \leq s(T) \leq s(K) = 3$.

CASE 2. *The orientations of \bar{l}_i and \bar{l}_j are inconsistent around the boundary of $\delta_i \cap \delta_j$.*

— In this case, only three crossings of $\bar{l}_i \cup \bar{l}_j$ are required to draw a contradiction.

FIGURE 7. \bar{l}_i and \bar{l}_j with at least four crossings.

For any crossing point Z of $\bar{l}_i \cup \bar{l}_j$, let Z_0 be the middle point of the two points $Z_i = \pi^{-1}(Z) \cap l_i$ and $Z_j = \pi^{-1}(Z) \cap l_j$. Let O be a point on \mathcal{Q} located so as to separate the four points of $K \cap \mathcal{Q}$ two by two. This case breaks into two subcases:

SUBCASE 2.1. *The closure of $l_i \cup l_j$ is not connected.* — Let A and B denote the starting point and the end point of the oriented arc l_i , respectively, and let C and D denote the starting point and the end point of the oriented arc l_j , respectively. Notice that A, B, C, D are the four points of $K \cap \mathcal{Q}$.

Suppose that O separates A and D . Then it also separates B and C . Let X and Y be the first and second crossing of $\bar{l}_i \cup \bar{l}_j$ along \bar{l}_i , respectively. Then the three parallel lines \mathcal{Q} , $\pi^{-1}(X)$ and $\pi^{-1}(Y)$ cut K into eight disjoint arcs. Consider the plane \mathcal{E} determined by the three points O , X_0 and Y_0 . Then each of the following six arcs, two from $K - (l_i \cup l_j)$, the two between \mathcal{Q} and $\pi^{-1}(X)$, and the two between $\pi^{-1}(X)$ and $\pi^{-1}(Y)$, crosses \mathcal{E} because the end points are separated by the plane. If O separates A and B , then it also separates C and D . In this case, each of the remaining two arcs between $\pi^{-1}(Y)$ and \mathcal{Q} crosses \mathcal{E} . If O does not separate A and B then it does not separate C and D either. In this case, the existence of the crossing next to Y guarantees that the union of the two remaining arcs between $\pi^{-1}(Y)$ and \mathcal{Q} crosses \mathcal{E} at least twice. Consequently, the knot K crosses \mathcal{E} at least eight times, resulting a contradiction $s(K) \geq 4$.

Suppose that O does not separate A and D . Then it does not separate B and C either. Then, according to Sublemma 5, one of the two arcs of $K - (l_i \cup l_j)$ which corresponds to a simple loop in Figure 4, together with the segment on \mathcal{Q} between its end points, must bound an embedded disk whose interior does not meet K . According to [10, Lemma 13], this kind of *topological triviality* can be avoided at the beginning when we choose the quadrisecant \mathcal{Q} . We now assume that our quadrisecant \mathcal{Q} is topologically nontrivial.

The quadrisecant \mathcal{Q} of a nontrivial knot K is defined to be *topologically nontrivial* if, for any two points P_1, P_2 of $K \cap \mathcal{Q}$ which are adjacent along \mathcal{Q} , any disk (possibly singular) bounded by the line segment $P_1 P_2$ and the arc of $K - \mathcal{Q}$ whose end points are P_1 and P_2 meets K in its interior.

SUBCASE 2.2. *The closure of $l_i \cup l_j$ is connected.* — We may assume the starting point of l_j is the end point of l_i . Let A and B denote the starting point and the end point of the oriented arc l_i , respectively, and C the the end point of the oriented arc l_j . Notice that A, B, C are three points of $K \cap \mathcal{Q}$. Let D be the remaining point of $K \cap \mathcal{Q}$.

Suppose O separates B from the two points A and C . Then B and D are on the same half-line of $\mathcal{Q} - O$. Let X and Y be the first and second crossing point of $\bar{l}_i \cup \bar{l}_j$ along \bar{l}_i . We choose two points $X_+ \in \pi^{-1}(X)$ and $Y_- \in \pi^{-1}(Y)$ so that

$$\overrightarrow{OX_+} = \overrightarrow{OX_0} + \frac{\|\overrightarrow{X_i X_j}\|}{\|\overrightarrow{OA}\|} \overrightarrow{OA} \quad \text{and} \quad \overrightarrow{OY_-} = \overrightarrow{OY_0} + \frac{\|\overrightarrow{Y_i Y_j}\|}{\|\overrightarrow{OB}\|} \overrightarrow{OB}.$$

Let \mathcal{E} be the plane determined by the three points O , X_+ and Y_- . Then each of the eight arcs in $K - (\pi^{-1}(X) \cup \pi^{-1}(Y) \cup \mathcal{Q})$ crosses \mathcal{E} because the end points are separated by the plane. So we get a contradiction $s(K) \geq 4$.

Suppose O separates A from the two points B and C . Then A and D are on the same half-line of $\mathcal{Q} - O$. Let l_a be the component of $K - \mathcal{Q}$ whose end points are at

A and D , and let l_b be the one whose end points are at C and D . By the assumption that \mathcal{Q} is topologically nontrivial, we know that \bar{l}_b is the only arc corresponding to a simple loop of Figure 4. Therefore there is a crossing point Y between \bar{l}_a and $\bar{l}_i \cup \bar{l}_j$. Consider the simple loop in $\bar{l}_i \cup \bar{l}_j$ created by the last crossing point of $\bar{l}_i \cup \bar{l}_j$ along \bar{l}_i . By Sublemma 6, this loop must have a crossing with \bar{l}_a . Again, by the assumption that \mathcal{Q} is topologically nontrivial, Y can be chosen so that the two vectors $\overrightarrow{Y_0Y_a}$ and \overrightarrow{OA} are in opposite directions. Let X be the first crossing point of $\bar{l}_i \cup \bar{l}_j$ along \bar{l}_i and let \mathcal{E} be the plane determined by the three points O , X_0 and Y_0 . Again we consider the eight disjoint arcs of $K - (\pi^{-1}(X) \cup \pi^{-1}(Y) \cup \mathcal{Q})$. Each of the following five, two from $l_a - \pi^{-1}(Y)$, the arc l_b , and the two between \mathcal{Q} and $\pi^{-1}(X)$, crosses the plane \mathcal{E} . It remains to check how many times the remaining three arcs cross \mathcal{E} . If \bar{l}_a crosses \bar{l}_i at Y , the three arcs joins the points X_i , Y_i , B , and X_j , successively. In this case, each of the three arcs crosses \mathcal{E} . If \bar{l}_a crosses \bar{l}_j at Y , the three arcs joins the points X_i , B , Y_j and X_j , successively. In this case, the arc joining B and Y_j crosses \mathcal{E} and the existence of a crossing point in $\bar{l}_i \cup \bar{l}_j$ other than X guarantees that the union of the two remaining arcs crosses the plane at least twice. Consequently, K crosses \mathcal{E} at least eight times, resulting a contradiction $s(K) \geq 4$.

The case when O separates C from the two points A and C can be handled similarly. \square

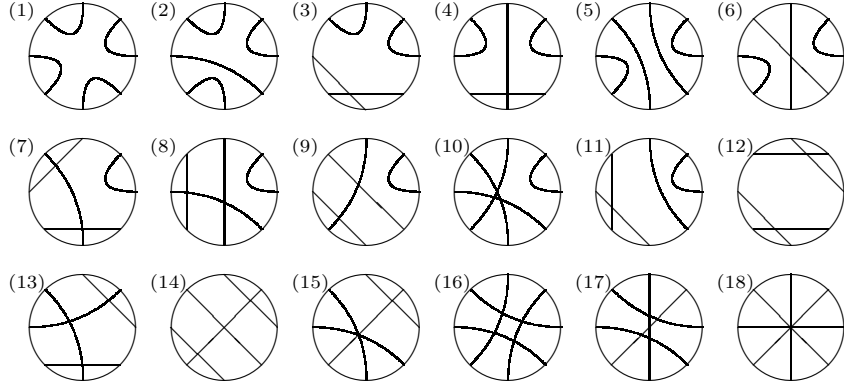


FIGURE 8. Patterns near $\pi(\mathcal{Q})$

Sublemma 8. *There are finitely many possible diagrams for K obtained from $\pi(K)$ by perturbing near the quadriseccant \mathcal{Q} .*

PROOF: Sublemma 5 and Sublemma 7 leave only finitely many possible projections of K on \mathcal{Q}^\perp outside a small neighborhood of $\pi(\mathcal{Q})$ up to planar isotopies. On the other hand, the projection of a cylindrical neighborhood of a quadriseccant of an arbitrary knot has eighteen possible patterns as shown in Figure 8 where all four arcs are smooth, up to small perturbations and planar isotopies. For each pair of the projections outside and inside a neighborhood of $\pi(\mathcal{Q})$, there are only finitely many ways to combine them to obtain a projection whose singular points are only

transversal double points. For each double point, there are only two choices for crossings. Consequently there are only finitely many possible diagrams of K on \mathcal{Q}^\perp . \square

Since there are only finitely many possible diagrams, there are only finitely many possible knot types.

4. AN EXAMPLE

The knot shown in Figure 9(a) is a figure eight knot parametrized¹ by

$$\begin{aligned} x(t) &= 307 \cos^3 t + 5346 \sin t \cos^2 t - 2663 \cos^2 t \\ &\quad - 26 \sin t \cos t - 1142 \cos t - 1378 \sin t + 1280 \\ y(t) &= 6337 \cos^3 t + 191 \sin t \cos^2 t + 691 \cos^2 t \\ &\quad + 103 \sin t \cos t - 5021 \cos t - 1019 \sin t + 677 \\ z(t) &= 373 \cos^3 t - 3157 \sin t \cos^2 t - 4436 \cos^2 t \\ &\quad - 1029 \sin t \cos t + 50 \cos t + 910 \sin t + 2222 \end{aligned}$$

for $0 \leq t \leq 2\pi$. Since its harmonic degree [18] is 3, its superbridge number is 3. Figure 9(b) shows its projection into the xy -plane. Up to scaling and reparametrization, this knot can be perturbed to have a polynomial parametrization

$$\begin{aligned} x(t) &= (2t-1)(4t-1)(10t-1)(25t-16)(25t-21)(50t-9) \times \\ &\quad (386t^6 - 708t^5 - 201t^4 + 945t^3 - 383t^2 - \frac{42224361}{1146679}t - \frac{2701080}{1146679}) \\ y(t) &= -70(2t-1)^2(4t-1)(10t-1)(25t-21)^2 \times \\ &\quad (229t^6 - 776t^5 + 806t^4 - 197t^3 - 56t^2 - \frac{1667040}{277477}t - \frac{104544}{277477}) \\ z(t) &= (20t-3)(25t-9)(25t-16)(25t-23)(1233t^8 - 5985t^7 + 11394t^6 \\ &\quad - 10375t^5 + 4167t^4 - 243t^3 - 179t^2 - \frac{2145804}{166595}t - \frac{712368}{832975}) \end{aligned}$$

for $0 \leq t \leq 1$. Figure 10 illustrates this knot. It is clear that the z -axis is a quadrisecant of this knot.² It is a combination of Figure 4(b) and Figure 8(17). Figure 9 and Figure 10 are of scale 1 : 1000.

5. PRIME KNOTS UP TO 9 CROSSINGS

All 3-superbridge knots are among the forty seven knots which are 2-bridge knots up to 9 crossings except the torus knots of types (2, 5), (2, 7) and (2, 9). They are marked with \star or \times in Table 1. The symbols in the first column are as in [1, 15]. The number 47 is a very rough upper bound for the number of 3-superbridge knots. To show that it is an upper bound, we only need to show that 3-superbridge knots cannot have minimal crossing number bigger than 9.

¹This parametrization was obtained by modifying Trautwein's parametrization in [18]:

$$\begin{aligned} x(t) &= 32 \cos t - 51 \sin t - 104 \cos 2t - 34 \sin 2t + 104 \cos 3t - 91 \sin 3t \\ y(t) &= 94 \cos t + 41 \sin t + 113 \cos 2t - 68 \cos 3t - 124 \sin 3t \\ z(t) &= 16 \cos t + 73 \sin t - 211 \cos 2t - 39 \sin 2t - 99 \cos 3t - 21 \sin 3t \end{aligned}$$

² This knot may have superbridge number bigger than 3. However, it can be reduced to 3 again, by applying Lemma 3 away from the z -axis.

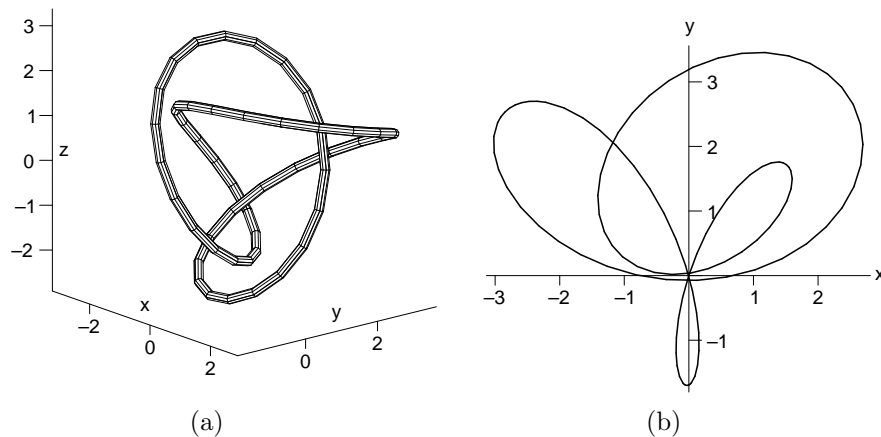
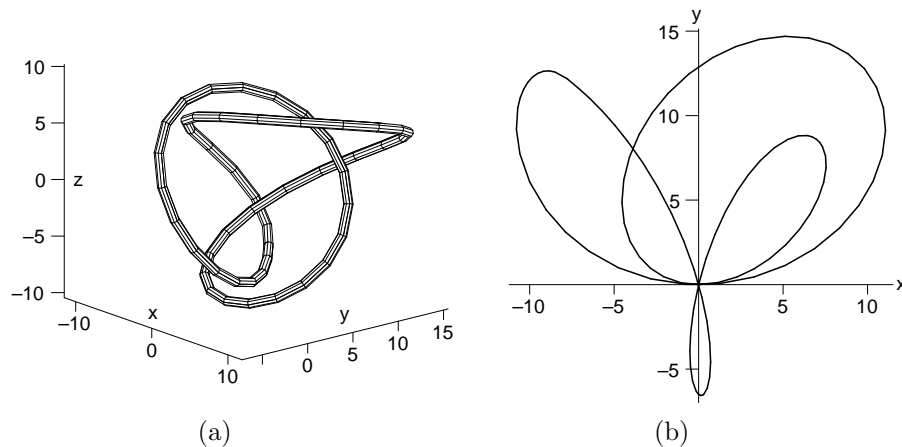


FIGURE 9. A figure eight knot with harmonic degree 3

FIGURE 10. A figure eight knot having z -axis as its quadrisecant

By Sublemma 7, we know that there are at most three double points in Figure 4(a) and at most six in Figure 4(b). After a little perturbation if necessary, each pattern in Figure 8 have at most six crossings. Therefore knot diagrams obtained from Figure 4(a) cannot have more than nine crossings. Since each pattern in Figure 8(1)–(14) has at most three crossings, knot diagrams obtained by any combination of Figure 4(b) and one of Figure 8(1)–(14) cannot have more than nine crossings. It remains to handle the combinations of Figure 4(b) and one of Figure 8(15)–(18). Since the quadrisecant \mathcal{Q} meets the knot K at four distinct points, the four arcs in any of Figure 8(15)–(18) are in distinct vertical levels. Therefore the crossings obtained from Figure 8(15)–(18) are not alternating, and hence any combination of Figure 4(b) and one of Figure 8(15)–(16) gives a non-alternating diagram of at most ten crossings. Because 3-superbridge knots are alternating knots, those obtained from such combinations must have minimal crossing number

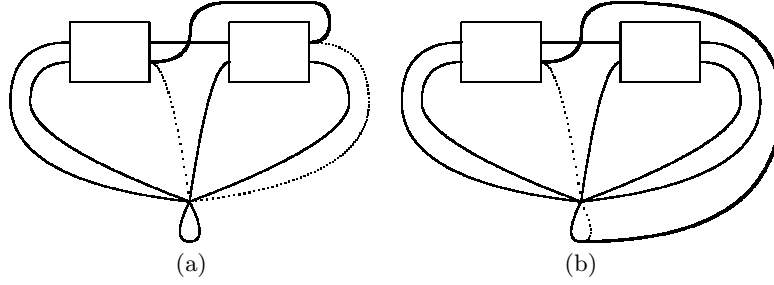


FIGURE 11. Moves reducing the number of crossings

at most 9. Now it remains to consider the combinations of Figure 4(b) and one of Figure 8(17)–(18). For any such combinations, we are able to move the uppermost arc or the lowermost arc at the quadruple point to reduce the number of crossings by one or two as needed to reduce the number down to at most 10. Two examples of such moves are shown in Figure 11. The resulting diagrams are still nonalternating, hence their minimal crossing numbers are at most 9.

Among all possible combinations of one of Figure 4 and one of Figure 8, we do not get all the forty seven knots mentioned above. There are 35 knots which either do not appear during the construction or are excluded by using the methods used in the proof of Theorem 2. They are marked with \times in the table. The remaining 12 knots are marked with \star . This list of 12 knots contains all the 3-superbridge knots. Details behind the selection of 12 knots and the rejection of 35 knots will be handled in [6].

K	$s[K]$		K	$s[K]$		K	$s[K]$		K	$s[K]$		K	$s[K]$	
3 ₁	3	○★	8 ₄	3–5	★	8 ₂₁	4	◇	9 ₁₇	4–7	×	9 ₃₄	4–6	◇
4 ₁	3	★	8 ₅	4–6	◇	9 ₁	4	○	9 ₁₈	4–6	×	9 ₃₅	4–6	◇
5 ₁	4	○	8 ₆	4–6	×	9 ₂	4–7	×	9 ₁₉	4–6	×	9 ₃₆	4–7	◇
5 ₂	3–4	★	8 ₇	3–6	★	9 ₃	4–6	×	9 ₂₀	4–6	×	9 ₃₇	4–7	◇
6 ₁	3–4	★	8 ₈	4–5	×	9 ₄	4–7	×	9 ₂₁	4–7	×	9 ₃₈	4–7	◇
6 ₂	3–4	★	8 ₉	3–6	★	9 ₅	4–6	×	9 ₂₂	4–7	◇	9 ₃₉	4–6	◇
6 ₃	3–4	★	8 ₁₀	4–6	◇	9 ₆	4–6	×	9 ₂₃	4–7	×	9 ₄₀	4	◇
7 ₁	4	○	8 ₁₁	4–5	×	9 ₇	4–6	×	9 ₂₄	4–6	◇	9 ₄₁	4	◇
7 ₂	3–4	★	8 ₁₂	4–6	×	9 ₈	4–6	×	9 ₂₅	4–7	◇	9 ₄₂	4	◇
7 ₃	3–4	★	8 ₁₃	4–5	×	9 ₉	4–6	×	9 ₂₆	4–6	×	9 ₄₃	4–5	◇
7 ₄	3–4	★	8 ₁₄	4–5	×	9 ₁₀	4–6	×	9 ₂₇	4–6	×	9 ₄₄	4–5	◇
7 ₅	4	×	8 ₁₅	4–6	◇	9 ₁₁	4–6	×	9 ₂₈	4–6	◇	9 ₄₅	4–5	◇
7 ₆	4	×	8 ₁₆	4	◇	9 ₁₂	4–6	×	9 ₂₉	4–7	◇	9 ₄₆	4	◇
7 ₇	4	×	8 ₁₇	4	◇	9 ₁₃	4–6	×	9 ₃₀	4–6	◇	9 ₄₇	4–6	◇
8 ₁	4–5	×	8 ₁₈	4	◇	9 ₁₄	4–7	×	9 ₃₁	4–6	×	9 ₄₈	4–6	◇
8 ₂	4–5	×	8 ₁₉	4	○○	9 ₁₅	4–5	×	9 ₃₂	4–6	◇	9 ₄₉	4–5	◇
8 ₃	4–6	×	8 ₂₀	4	◇	9 ₁₆	4–7	◇	9 ₃₃	4–6	◇			

TABLE 1. Superbridge index of prime knots up to 9 crossings

In the table, torus knots are marked with \circ , for which the superbridge index is determined by Theorem 1.

If a knot is presented as a polygon in space, one half of the number of edges is an upper bound of the superbridge index [7]. The number or the upper limit of the range of numbers in the second column of the table is the largest integer not exceeding one half of the minimal edge number or the best-known minimal edge number [2, 3, 7, 11]. For the five knots, 3_1 , 4_1 , 5_1 , 5_2 and 8_{19} , this number is equal to the harmonic degree [18]. It is known that 2-bridge knots cannot have superbridge index bigger than seven [5].

On the other hand, for those marked with \star or \diamond , the number or the lower limit of the range of numbers in the second column is one bigger than the bridge index. For those marked with \times or only with \circ , the number or the lower limit of the range of numbers in the second column is two bigger than the bridge index.

Among the 18 knots whose superbridge index is determined in the table, only 7_5 , 7_6 and 7_7 were newly found by this work.

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KOREA ADVANCED INSTITUTE OF SCIENCE AND TECHNOLOGY

E-mail address: jcbout@math.kaist.ac.kr, trefoil@kaist.ac.kr